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Capillary–gravity waves of solitary type on deep water

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On physical grounds it was recently suggested that limiting capillary–gravity waves of solitary type may exist on the surface of deep water (Longuet-Higgins 1988). This paper describes accurate numerical calculations which support the conjecture. The limiting wave has a phase speed $c = 0.9267\ (g\tau)^{\frac{1}{4}}$. It is one of a family of solitary waves having speeds $c \leqslant 1.30\ (g\tau)^{\frac{1}{4}}$. The maximum angle of inclination $\alpha_{\max}$ of the free surface is a monotonically decreasing function of the speed c. Physical arguments suggest that $\alpha_{\max}$ has a positive lower bound.

1. Introduction

Solitary waves in shallow water have been recognised since the classical observations by Scott Russell (1838) and the approximate theories by Boussinesq (1871) and Rayleigh (1876). For reviews of modern numerical calculations of this type of wave see Miles (1980) and Schwartz & Fenton (1982).

Solitary-type solutions for the *envelope* of a group of waves on deep water – that it so say solutions of the approximate nonlinear Schrödinger equation – have been found by Chou & Mei (1970, 1971) and others; for observations see Yuen & Lake (1975).

Solitary waves can also exist on the stratified region between two fluids of different densities, unbounded above and below; see Benjamin (1967). They were observed experimentally by Davis & Acrivos (1967).

Solitary capillary–gravity waves in water of *finite* depth have also been calculated by Korteweg & de Vries (1985) and by Hunter & Vanden-Broeck (1983).

Until very recently, no such phenomenon was suspected for capillary or capillary–gravity waves on deep water. The well-known linear theory for capillary–gravity (C–G) waves of small steepness (Lamb 1932) yields of course only periodic solutions, with finite wavelength. Wilton (1915) took into account some nonlinear terms and found that more than one type of periodic C–G wave with given length could sometimes exist. Accurate numerical calculations for C–G waves of finite steepness have been published by Bloor (1978), Schwartz & Vanden-Broeck (1979), Chen & Saffman (1979) and Hogan (1980, 1981) among others, but without any indication of a limiting form corresponding to waves of infinite length. Nonlinear C–G waves were also observed in the laboratory by Schooley (1958).

Earlier, Crapper (1957) had found an exact expression for pure capillary periodic waves on deep water. His analytic solution showed clearly how, as the wave steepness is increased, the wave crests become more rounded and the wave troughs more narrow until finally a limiting form is reached in which adjacent wave crests touch one another and pinch off a pocket of 'air' in the trough. At the same time the particle velocities in the wave crest become very small (in a frame moving horizontally with the wave speed). Hence the wave crests are almost in static equilibrium, the particle speed having little influence on the surface boundary condition.

In a recent paper Longuet-Higgins (1988) pointed out that by applying a similar physical assumption to the profiles of the more general C–G waves one could formulate a solvable equation for the profiles of the wave crests. The solution suggested that solitary waves of this type should exist. In fact by patching the limiting wave crest to an appropriate expression for the wave troughs, expressions were obtained for both the speed and the profile of limiting C–G waves (of finite length) which agreed fairly well with the values obtained by the accurate calculations of Schwartz & Vanden-Broeck (1979), Chen & Saffman (1979) and Hogan (1980, 1981). Furthermore, the argument suggested the existence of a solitary C–G wave of limiting form, i.e. one enclosing a 'pocket of air'.

The purpose of the present paper is to investigate by an accurate numerical method whether such a solitary wave exists or not. The mathematical problem is formulated below in §2, where equations are derived for the surface slope α as a function of the velocity potential $(\phi+i\psi)$. In §3 we describe a method of calculation by means of Fourier expansion in a transformed plane, the form of the solution being similar to that suggested by the physical argument in Longuet-Higgins (1988).

The results are described in §4. It appears that a solitary wave of limiting form does indeed exist – its phase-speed is $0.9276\,(g\tau)^{\frac{1}{4}}$, where g denotes gravity and τ surface tension. Moreover it is one member of a whole family of solitary waves, whose speed c depends upon the maximum slope α of the surface profile. In dimensionless units ($g=\tau=1$), we find convergent solutions for c lying in the range $0.9276 \leqslant c \leqslant 1.30$.

The behaviour of the velocity field at infinity in the horizontal direction is not exponential, as in shallow-water waves, but instead algebraic, the surface displacement η falling off like x^{-2}, where x is horizontal distance. (An inner scale is exponential, however). In this respect the C–G waves, though differing from solitary waves in shallow water, resemble both the 'envelope solitons' on the surface of deep water and the internal solitary waves on the interface of two fluids unbounded above and below.

The question whether solitary C–G waves can exist for arbitarily small values of the maximum surface slope $\alpha_{\max}$ is discussed in §5. In §§6–8 we consider the flow relative to stationary axes, and evaluate some of its integral properties. The net displaced mass in each wave is zero, so that no circulation at infinity is required to counteract the additional buoyancy.

2. Formulation of the problem

We consider a steady, irrotational wave, travelling with speed c to the left, on the surface of a frictionless, incompressible fluid of infinite depth, as in figure 1. Seen in a frame of reference travelling with speed c to the left, the wave appears as a steady flow, in which the velocity at infinite depth is a horizontal flow with speed c to the right.

Let ϕ and ψ denote the velocity potential and stream function, and x, y denote

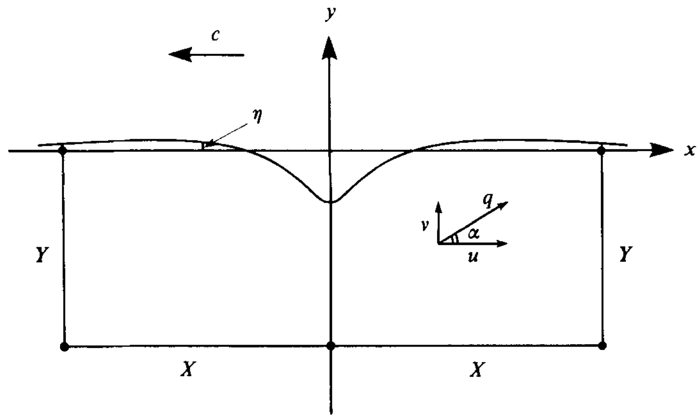

FIGURE 1. Definition diagram showing coordinates for a solitary wave. The direction of wave propagation is to the left.

horizontal and vertical coordinates as shown. Then the magnitude q and direction α of the velocity at any point in the flow are given by

$$q\,\mathrm{e}^{-\mathrm{i}\alpha} = c\frac{\mathrm{d}w}{\mathrm{d}z}, \tag{2.1}$$

where $w = (\phi + \mathrm{i}\psi)/c$ and $z = x + \mathrm{i}y$. Hence if we define

$$\beta = \ln(q/c) \tag{2.2}$$

we see that

$$\alpha + \mathrm{i}\beta = \mathrm{i}(\beta - \mathrm{i}\alpha) = \mathrm{i}\ln\frac{\mathrm{d}w}{\mathrm{d}z}. \tag{2.3}$$

This is an analytic function of w or of z, vanishing as y and ψ tend to $-\infty$.

Consider now the boundary condition at the free surface. It will be convenient to choose units of mass, length and time in which the density ρ, the acceleration of gravity g and the surface-tension constant τ are all unity. (In dimensional terms, the units of length and velocity are $(\tau/\rho g)^{\frac{1}{2}}$ and $(\tau g/\rho)^{\frac{1}{4}}$ respectively.) Then the condition of constant pressure at the free surface $\psi = 0$ becomes simply

$$y + \tfrac{1}{2}q^2 + \kappa = \text{constant}, \tag{2.4}$$

where κ denotes the curvature. If s denotes arclength measured along the surface we choose the sign of κ to be such that

$$\kappa = -\frac{\mathrm{d}\alpha}{\mathrm{d}s} = -q\frac{\mathrm{d}\alpha}{\mathrm{d}\phi}, \tag{2.5}$$

i.e. κ is positive when the surface is convex upwards. Now we also have

$$\frac{\mathrm{d}y}{\mathrm{d}\phi} = \frac{1}{q}\frac{\mathrm{d}y}{\mathrm{d}s} = \frac{1}{q}\sin\alpha \tag{2.6}$$

so that on differentiating (2.4) with respect to ϕ along the free surface we have

$$\frac{1}{q}\sin\alpha + q\frac{\mathrm{d}q}{\mathrm{d}\phi} - \frac{\mathrm{d}}{\mathrm{d}\phi}\left(q\frac{\mathrm{d}\alpha}{\mathrm{d}\phi}\right) = 0. \tag{2.7}$$

On substitution for q from (2.2) this becomes

$$\frac{1}{c}\mathrm{e}^{-\beta}\sin\alpha+c^2\,\mathrm{e}^{2\beta}\frac{\mathrm{d}\beta}{\mathrm{d}\phi}-c\,\mathrm{e}^{\beta}\left(\frac{\mathrm{d}\alpha}{\mathrm{d}\phi}\frac{\mathrm{d}\beta}{\mathrm{d}\phi}+\frac{\mathrm{d}^2\alpha}{\mathrm{d}\phi^2}\right)=0. \tag{2.8}$$

At the same time α and β are conjugate functions of ϕ, so that β is the Hilbert transform of α (see Bloor 1978).

We wish to find solutions to (2.8) representing solitary waves, i.e. such that $(\alpha+\mathrm{i}\beta)$ tends to zero as $|x|$ or $|\phi|$ tends to infinity. In that case $q\rightarrow c$ and $\kappa\rightarrow 0$ at infinity. Hence if we choose the origin of y so as to make $y\rightarrow 0$ at inifinity, (2.4) becomes simply

$$y+\tfrac{1}{2}q^2+\kappa=\tfrac{1}{2}c^2. \tag{2.9}$$

However, because y is related to α through its derivative $\mathrm{d}y/\mathrm{d}\phi$ (see (2.6)) it is more convenient to deal with the differentiated form of the boundary condition, i.e. (2.8).

For computation, it is most convenient to multiply (2.8) by $c\,\mathrm{e}^{\beta}$ and to take as independent variable $\theta=\phi/c$. Then (2.8) becomes

$$\sin\alpha+c^2\beta'\,\mathrm{e}^{3\beta}-(\alpha''+\alpha'\beta')\,\mathrm{e}^{2\beta}=0 \tag{2.10}$$

in which a prime denotes $\mathrm{d}/\mathrm{d}\theta$.

3. Approach to a solution

Consider the velocity field at infinity. It is clear that the vortex model adopted earlier (Longuet-Higgins 1988), which gave a rough approximation to the phase-speed c of limiting waves, cannot be exact. For, if the vertical component of velocity at the free surface varied as x^{-1} at infinity then the vertical displacement would vary as $\ln x$, which would make the potential energy infinite (the kinetic energy also). We must therefore suppose that the vertical velocity vanishes more rapidly than x^{-1}. Hence writing

$$\mathrm{i}(\alpha+\mathrm{i}\beta)=F \tag{3.1}$$

we seek solutions in the form

$$F(w)=(w-\mathrm{i}a)^{-2}\,G(w), \tag{3.2}$$

where $w=(\phi+\mathrm{i}\psi)/c$ as before and a is a positive, real constant. We assume that $G(w)$ is analytic and bounded in the half-plane $\psi\leqslant 0$, that is $\operatorname{Im}(w)\leqslant 0$.

We now transform the half-plane $\operatorname{Im}(w)<0$ into the interior of the unit circle $|\zeta|=1$ by writing

$$\zeta=\frac{ib+w}{ib-w},\quad w=ib\frac{1-\zeta}{1+\zeta}, \tag{3.3}$$

where b is another positive, real constant; see figure 2. $G(w)$ is analytic and bounded everywhere inside the circle $|\zeta|=1$, so we may expand G in a power series:

$$G(w)=A_1-A_2\,\zeta+A_3\,\zeta^2-\ldots, \tag{3.4}$$

where the coefficients A_n are to be determined. If these are real, the wave profile will be symmetric about the line $x=0$.

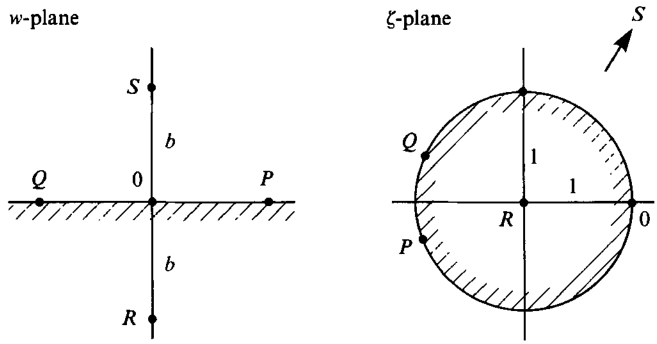

FIGURE 2. Transformation from the w-plane to the ζ-plane by equations (3.3).

The boundary condition (2.10) is expressed in terms of

$$\alpha = \operatorname{Im}(F), \qquad \beta = -\operatorname{Re}(F) \tag{3.5}$$

and

$$\alpha' = \operatorname{Im}\left(\frac{dF}{dw}\right) \qquad \beta' = -\operatorname{Re}\left(\frac{dF}{dw}\right) \tag{3.6}$$

etc. From (3.2) we have

$$\frac{dF}{dw} = \frac{-2G}{(w-ia)^3} + \frac{1}{(w-ia)^2}\frac{dG}{dw} \tag{3.7}$$

and

$$\frac{d^2F}{dw^2} = \frac{6G}{(w-ia)^4} - \frac{4}{(w-ia)^3}\frac{dG}{dw} + \frac{1}{(w-ia)^2}\frac{d^2G}{dw^2}. \tag{3.8}$$

Also from (3.3)

$$\frac{dG}{dw} = \frac{2ib}{(w-ib)^2}\frac{dG}{d\zeta} \tag{3.9}$$

and

$$\frac{d^2G}{dw^2} = \frac{-4ib}{(w-ib)^3}\frac{dG}{d\zeta} - \frac{4b^2}{(w-ib)^4}\frac{d^2G}{d\zeta^2}, \tag{3.10}$$

$dG/d\zeta$ and $d^2G/d\zeta^2$ being given in terms of the coefficients A_n through (3.4) and its derivatives.

The boundary condition (2.10) is clearly nonlinear in the coefficients A_n. However, we may apply a collocation method. Thus we truncate the series (3.4) at a finite value N of n and choose $A_1 \dots A_N$ so as to make the left-hand side of (2.10) vanish at N suitably spaced values of ϕ in the range $(0, \infty)$. These may be chosen so as to cover the range densely in the limit $N \to \infty$. Provided the resulting process converges we have a numerical solution.

In that case the coordinates (x, y) of the free surface may be calculated from

$$\left.\begin{aligned} x &= \int_0^s \cos\alpha \, ds = \int_0^\theta e^{-\beta}\cos\alpha \, d\theta, \\ y &= \int_0^s \sin\alpha \, ds = \int_0^\theta e^{-\beta}\sin\alpha \, d\theta. \end{aligned}\right\} \tag{3.11}$$

Since α and β are odd and even functions, respectively, of θ, the boundary condition (2.10) is automatically satisfied at $\theta = 0$. For the remaining collocation points it is convenient to take

$$\theta_m = h \tan \frac{(2m-1)\pi}{4N}, \quad m = 1, 2, \ldots, N, \tag{3.12}$$

where h is a suitable constant. (When $h = b$, the points are evenly spaced around the circle $|\zeta| = 1$.)

4. Numerical results

It was found convenient to take $a = 1.0$, then $0.9 \leqslant b \leqslant 1 \cdot 1$ and $0.7 \leqslant h \leqslant 0.9$. With these values it was found that the solutions, found by Newton's method, converged to limits which were independent of the starting values of the coefficients A_n within sensible limits, and also were independent of a, b and h, provided that the phase speed c lay in the range $0.90 < c \leqslant 1.30$. The surface profiles corresponding to values of c in this range are shown in figures 3(*a*) to 3(*f*). The profile for $c = 0.9276$ in figure 3(*a*) represents the critical case when the free surface touches itself and encloses a 'bubble', as first found by Crapper (1957) for pure capillary waves. For values of c less than 0.9276 the computed solitary wave profiles are self-intersecting and so do not represent possible physical situations.

It will be seen that all of the profiles are of the same general type as suggested by the physical reasoning in Longuet-Higgins (1988), that is, they have sharply curved troughs, then rise to a maximum, before approaching the free surface from above. The limiting case $c = 0.9276$ is very similar to the approximation shown in figure 15 of Longuet-Higgins (1988).

Details of the solutions are given in table 1. In the third column of table 1 is shown the Bernoulli constant

$$B = (y - y_0) + \tfrac{1}{2}(q^2 - c^2) + \kappa \tag{4.1}$$

where y_0 denotes $y_{\theta=0}$. If the solution were exact we should have $B = -y_0$. In the next column, ΔB denotes the maximum departure of B from its stated value over the whole range $-10 \leqslant \theta \leqslant 10$. The remaining columns show the values of $q_{\max}$, $|\kappa_{\max}|$, $\alpha_{\max}$ and $y_{\max}$, where $y_{\max}$ is the maximum elevation of the free surface; also shown are the horizontal coordinate x_m of this point and the total height $H = y_{\max} - y_0$.

A plot of the functions $\alpha(\theta)$ and $\beta(\theta)$ in the critical case $c = 0.9276$ is shown in figure 4. This shows that in spite of the large curvature in the wave trough ($\theta = 0$), both α and β are quite smooth functions when plotted as functions of the velocity potential $c\theta$.

Some numerical values of the coefficients A_n when $a = b = 1$ are given in table 2. This shows that the sequence is dominated by the first coefficient A_1. If this alone is retained then we have from (3.2) the approximate expressions

$$\alpha(\theta) = A_1 \frac{2\theta}{(1+\theta^2)^2}, \qquad \beta(\theta) = A_1 \frac{1-\theta^2}{(1+\theta^2)^2}, \tag{4.2}$$

which may be used in rough work. However when $\theta > 0$ we see that the expression for α is always positive, whereas in figure 4 the function α changes sign at $\theta = 3.042$ when $x = 2.864$.

Figures 5 and 6 show plots of the maximum surface slope $\alpha_{\max}$ and curvature $|\kappa_{\max}|$ as functions of the phase speed c. From this it appears that both $\alpha_{\max}$ and

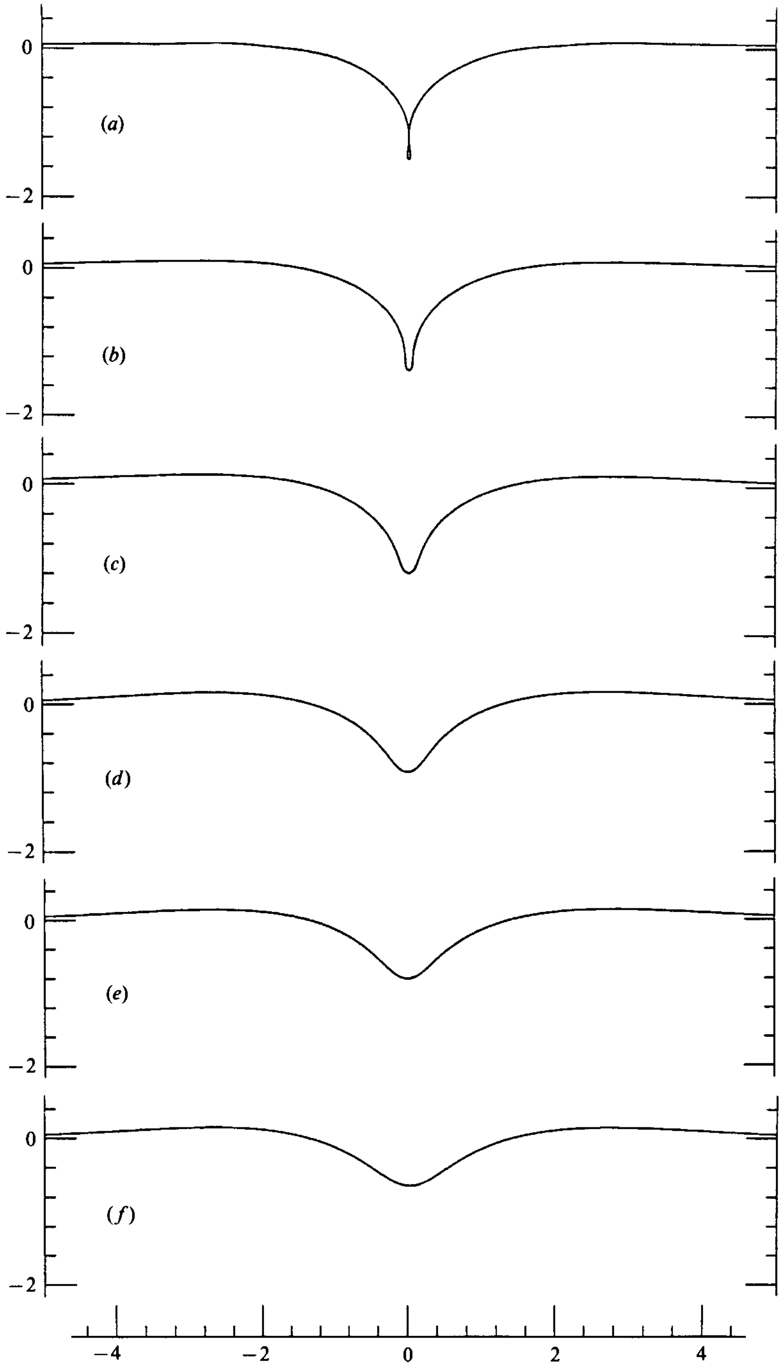

FIGURE 3. Profiles of computed solitary waves: (*a*) $c = 0.9276$, (*b*) $c = 1.00$, (*c*) $c = 1.10$, (*d*) $c = 1.20$, (*e*) $c = 1.25$, (*f*) $c = 1.30$.

c	N	B	ΔB	q_{max}	κ_0	α_{max}	y_{max}	x_m	H
0.9276	60	1.4863	.0001	11.83	68.10	1.712	.090	2.86	1.576
0.95	60	1.4515	.0001	10.83	56.81	1.652	.095	2.83	1.547
1.00	55	1.3675	.0001	8.905	37.78	1.514	.107	2.78	1.474
1.05	55	1.2745	.0001	7.345	25.17	1.371	.118	2.74	1.393
1.10	50	1.1717	.0006	6.077	16.69	1.224	.129	2.72	1.300
1.15	35	1.0583	.0009	5.032	10.94	1.072	.138	2.70	1.196
1.20	35	0.9336	.0012	4.167	7.03	0.917	.145	2.69	1.079
1.25	20	0.798	.005	3.45	4.36	0.76	.15	2.69	0.95
†1.30	20	0.645	.007	2.8	2.5	0.59	.14	2.7	0.79

† Derived from (4.3); see table 3.

TABLE 1. Parameters for the profiles of solitary waves

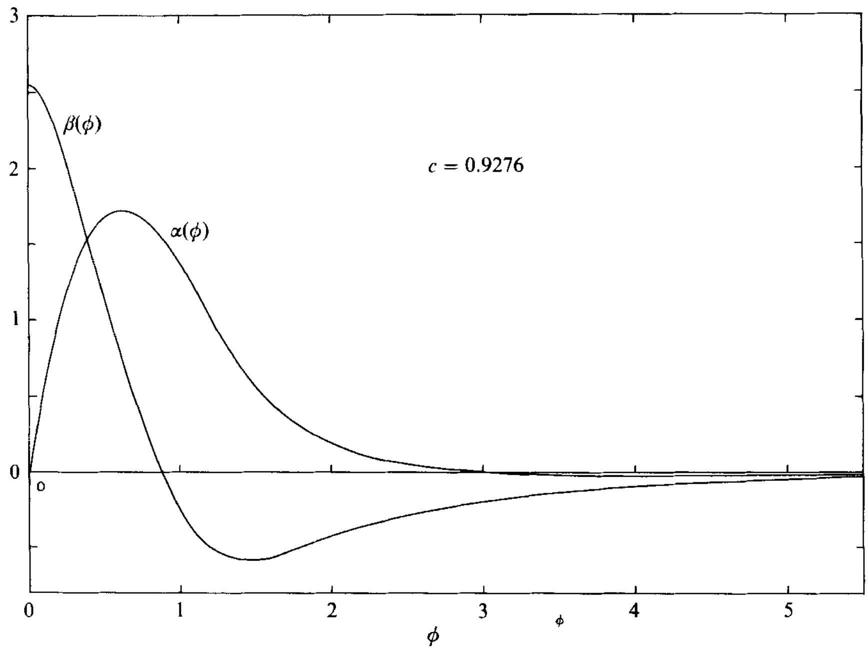

FIGURE 4. The functions $\alpha(\theta)$ and $\beta(\theta)$, giving the direction and magnitude of the velocity $c\,e^{\beta+i\alpha}$ at the free surface in the limiting case $c = 0.9276$.

$|\kappa_{max}|$ decrease monotonically with c, but provided $c \leqslant 1.30$ both remain positive. At values of c only slightly beyond 1.30 satisfactory convergence of the coefficients A_n, and hence of the solution, became impossible to obtain by the present method. The question of the existence of solutions beyond this point will be discussed in §5.

Here we mention that we carried out an alternative set of calculations starting with the less restrictive assumption

$$F(w) = \mathrm{i}(w-\mathrm{i}a)^{-1}\bar{G}(w) \tag{4.3}$$

instead of (3.2). The function $\bar{G}$ was expanded in a power series

$$\bar{G} = B_1 - B_2\zeta + B_3\zeta^2 - \dots \tag{4.4}$$

n	A_n	n	A_n	n	A_N
1	2.3508	16	−.0026	31	.0005
2	0.5359	17	−.0009	32	−.0007
3	−0.6316	18	.0032	33	.0008
4	0.3210	19	−.0045	34	−.0009
5	−0.0336	20	.0051	35	.0009
6	0.0083	21	−.0051	36	−.0009
7	−0.0106	22	.0048	37	.0008
8	−0.0058	23	−.0042	38	−.0007
9	0.0319	24	.0034	39	.0006
10	−0.0399	25	−.0027	40	−.0005
11	0.0344	26	.0019	41	.0004
12	−0.0255	27	−.0013	42	−.0003
13	0.0185	28	.0007	43	.0002
14	−0.0128	29	−.0002	44	−.0001
15	0.0074	30	−.0002	45	.0001

TABLE 2. Numerical values of the coefficients A_n in (3.4) when $c = 0.9276$ and $a = b = 1.0$

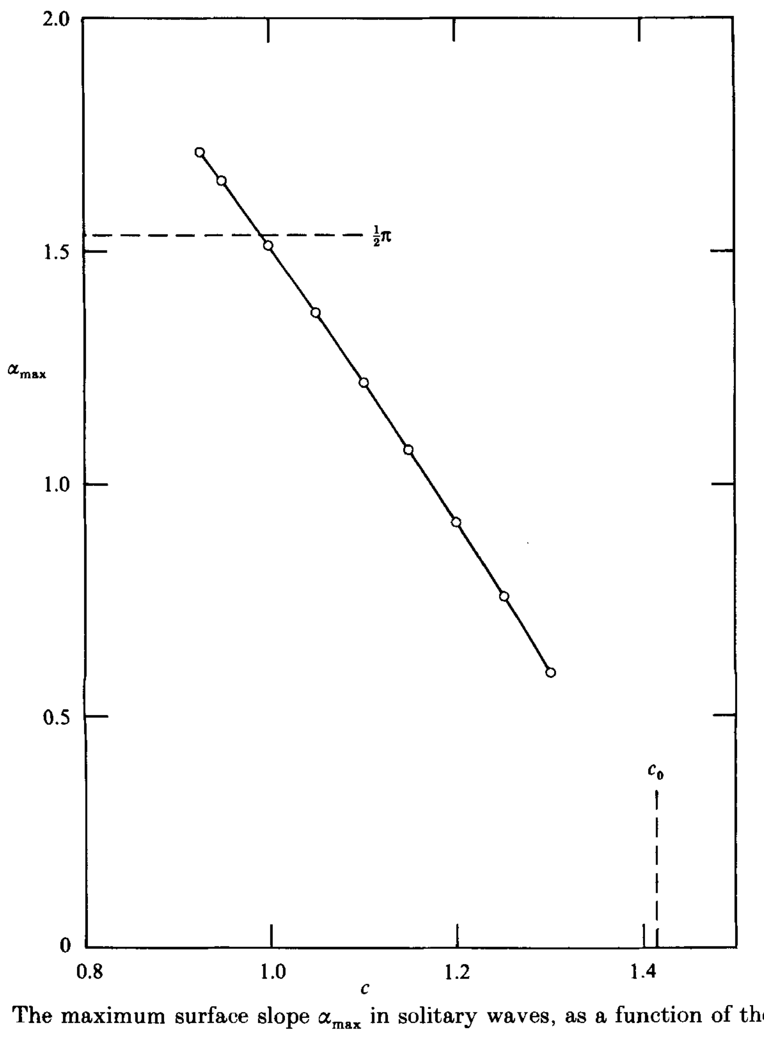

FIGURE 5. The maximum surface slope α_{max} in solitary waves, as a function of the phase speed c.

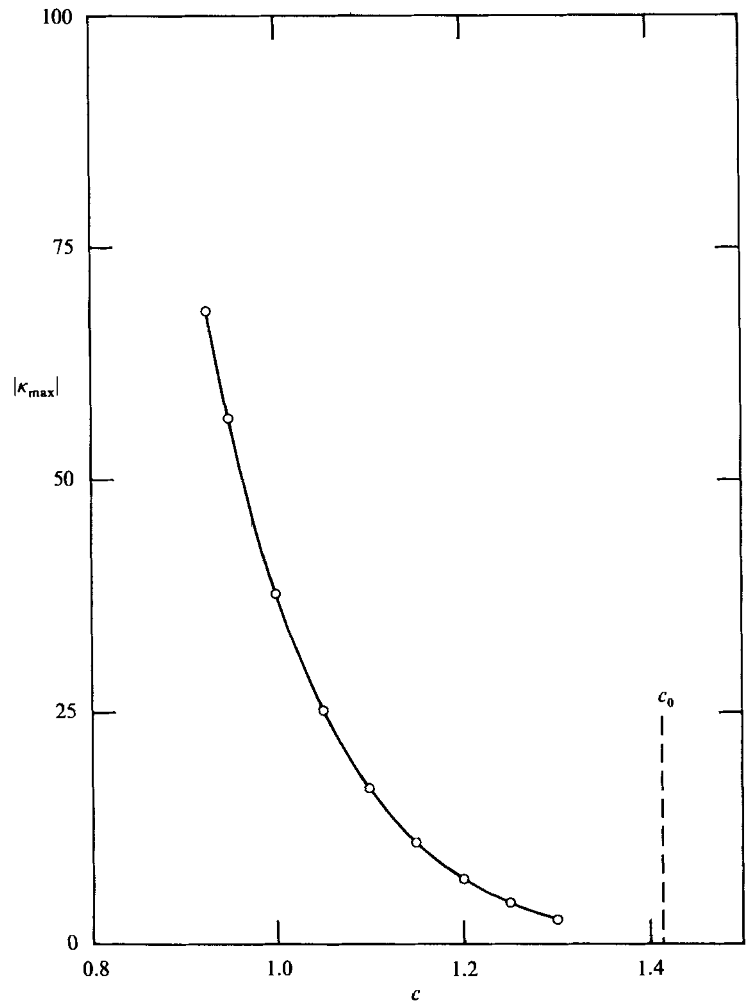

FIGURE 6. The curvature $|\kappa_{max}|$ in the wave trough as a function of the phase speed c.

similar to (3.4). The alternative form (4.3) allows for the possible presence of a vortex term at large values of $|w|$, when $\zeta \to -1$. By (3.3), the circulation in the vortex is proportional to

$$\bar{G}(\infty) = B_1 - B_2 + B_3 - \dots . \tag{4.5}$$

The results of this calculation were numerically indistinguishable from those obtained with the assumption (3.2), i.e. the two calculations converged to the same solution. This was explained by the fact that with increasing N the series (4.5) converged numerically to zero, showing that the strength of the vortex at infinity was indeed zero.

Nevertheless the *rate* of convergence of the coefficients B_n and of the corresponding Bernoulli constant B was better by a factor of 2 to 10 than the rate of convergence of the coefficients A_n. This can be seen from the numbers given in table 3. Thus if we disregard the far field, the assumption (4.3) yields significantly better results than (3.2).

Concerning the degree of accuracy, we note that the variation ΔB in the Bernoulli

c	N	B	ΔB
0.9276	50	1.48635	.00004
0.95	50	1.45154	.00005
1.00	45	1.36751	.00006
1.05	45	1.27445	.00008
1.10	45	1.17168	.00014
1.15	40	1.0582	.0008
1.20	40	0.9334	.0006
1.25	40	0.7959	.0017
1.30	20	0.645	.007

TABLE 3. Convergence obtained with equation (4.3)

constant should really be compared with the largest term in B, that is to say $\frac{1}{2}q^2_{max}$. From table 3 it appears that for the limiting wave ($c = 0.9276$), the ratio $2\Delta B/q^2_{max}$ is less than 10^{-6}. Thus it can hardly be doubted that a solution exists. Even when $c = 1.30$ the ratio is less than 2×10^{-3}. However, when $c \geqslant 1.10$ any increase in the number N of coefficients beyond the number stated does not appear to improve the accuracy of the solution, at least by this method.

5. Discussion: the lower bound of α_{max}

We have found strong evidence that waves of solitary type do exist within a certain range of the wave speed c and the slope amplitude α_{max}. A question that naturally arises is: what is the maximum value of c, and the corresponding value of α_{max}? In particular, is it possible that α_{max} can tend to zero?

Now in the parallel case of solitary waves in shallow water there do exist waves of arbitarily small slope, whose speed tends to the value $(gd)^{\frac{1}{2}}$ (d denotes the undisturbed depth). But in this limit the characteristic horizontal lengthscale of the wave becomes indefinitely large. The same is true also of other known types of solitary wave in water: the 'envelope soliton' and the internal solitary wave (Benjamin 1967). In the present case of capillary–gravity waves, to extend the characteristic lengthscale while simultaneously diminishing α_{max} would mean reducing the local curvature and hence decreasing the relative importance of surface tension. This would diminish the opportunity for nonlinear effects to produce a solitary wave.

Hence for C–G waves, the only possibility, at low values of α_{max}, would be for the horizontal scale of the wave to simultaneously decrease, in some limited part of the wave profile. This would imply a relatively narrow region of sharp curvature, presumably in the wave trough where most of the increase in $|\alpha|$ would take place. Outside this region the curvature would be even smaller.

On the contrary, we see from figures 5 and 6 that $\alpha_{max}/|\kappa_{max}|$ shows no sign of tending to zero as c increases beyond 1.30. This fact suggests that at the upper end of the range of c there is actually no concentration of curvature in the wave troughs – unlike the situation at the lower end of the range.

We must conclude, at least tentatively, that for C–G waves the lower bound for α_{max} is positive. Hence, the present nonlinear solutions are not analytically contiguous to the well-known periodic surface waves of low amplitude. On the other hand they almost certainly are contiguous to C–G waves of finite amplitude and relatively large wavelength such as have been investigated by previous authors.

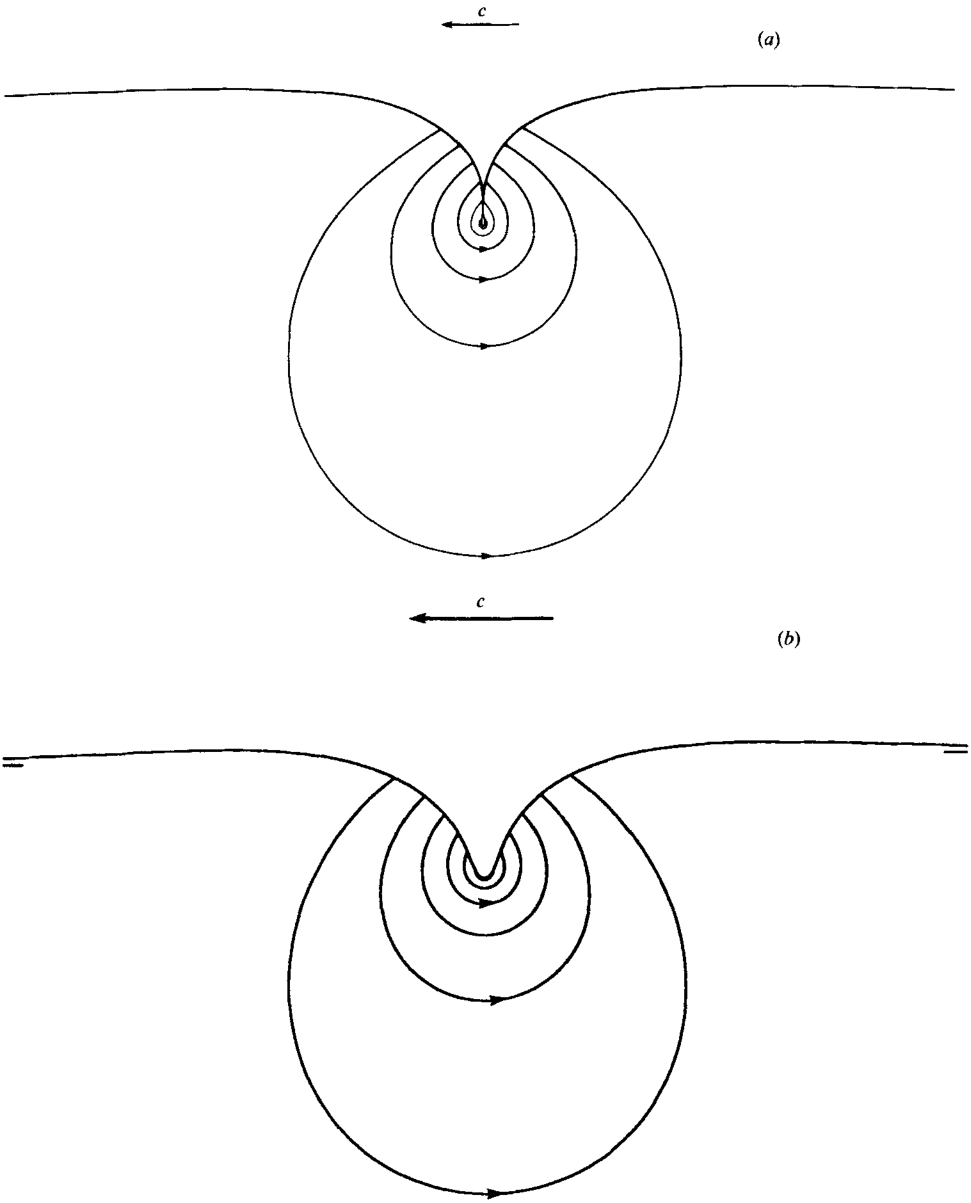

FIGURE 7. Streamlines of the flow as seen in a stationary frame of reference: (a) $c = 0.9276$, (b) $c = 1.10$.

6. The flow relative to deep water

Consider now the flow as seen in a stationary frame of reference, in which the particle velocity at infinite depth is zero. If χ' denotes the complex velocity potential in this frame we have

$$\chi' = (\phi + i\psi) - c(x+iy) = c(w-z) \tag{6.1}$$

and so

$$\frac{d\chi'}{dz} = c\left(\frac{dw}{dz} - 1\right) = c(e^{\beta - i\alpha} - 1) = c(e^{-F} - 1) \tag{6.2}$$

by (2.1), (2.2), (2.3) and (3.1). For large values of $|z|$ where α and β, hence F, are small, we have

$$\frac{\mathrm{d}\chi'}{\mathrm{d}z} \sim -cF. \tag{6.3}$$

Assuming that as $w \to \infty$

$$G(w) \to D/c \tag{6.4}$$

say, we have

$$F \sim \frac{D/c}{(w-\mathrm{i}a)^2} \sim \frac{D}{cz^2}. \tag{6.5}$$

From (6.3) and (6.5),

$$\frac{\mathrm{d}\chi'}{\mathrm{d}z} \sim -\frac{D}{z^2} \tag{6.6}$$

hence

$$\chi' \sim \frac{D}{z}. \tag{6.7}$$

Thus the far-field flow is approximately a dipole.

The streamlines in the relative flow are shown in figure 7, at various values of c. It can be seen how the velocity field resembles that due to a body near the free surface moving horizontally with speed c to the left. In this case, however, the net displacement of mass is zero. The impulse I, given by (8.11) comes entirely from the 'virtual mass' associated with the wave, as we shall now see.

7. Integral properties: the net displacement

In this and the following Section we shall consider some integral quantities of the motion relative to fixed axes. If $y = \eta$ denotes the elevation of the surface above the undisturbed level at infinity we may define the net displacement, or excess mass

$$M = \int_{-\infty}^{\infty} \eta \,\mathrm{d}x. \tag{7.1}$$

By analogy with shallow-water waves (Longuet-Higgins 1974) we may also define the 'circulation'

$$C = \int_{-\infty}^{\infty} (u-c)\,\mathrm{d}x = [\phi']_{-\infty}^{\infty} \tag{7.2}$$

where $\phi' = \phi - cx$.

We shall now prove a general theorem relating the excess mass M to the circulation C, namely

$$M = -cC. \tag{7.3}$$

For, in the steady flow relative to axes moving with speed c to the left consider the flux of vertical momentum into the region bounded by the free surface $y = \eta$, the horizontal plane $y = -Y$ and the two vertical planes $x \pm X$ (see figure 1). This flux is given by

$$\int_{-X}^{X} [(p+v^2)_{y=-Y} - p_{y=\eta}]\,\mathrm{d}x - \int_{-Y}^{\eta} uv\,\mathrm{d}y \bigg|_{x=-X}^{x=X}. \tag{7.4}$$

In a steady state this must be exactly balanced by the rate of increased vertical momentum due to the downwards action of gravity, namely

$$-\int_{-X}^{X} (Y+\eta)\,\mathrm{d}x. \tag{7.5}$$

Now from Bernoulli's equation for the interior of the fluid we have

$$p_{y=-Y}-p_{y=\eta} = [\tfrac{1}{2}(u^2+v^2)+y]_{y=-Y}^{y=\eta}. \tag{7.6}$$

So on equating (7.4) and (7.5) we obtain

$$\int_{-X}^{X} \tfrac{1}{2}(u^2+v^2)_{y=\eta}\,\mathrm{d}x = \int_{-Y}^{\eta} uv\,\mathrm{d}y\Big|_{-X}^{X} + \int_{-X}^{X} \tfrac{1}{2}(u^2-v^2)\,\mathrm{d}x. \tag{7.7}$$

But the boundary condition at the free surface can be written

$$\tfrac{1}{2}(u^2+v^2)_{y=\eta}+\eta+\kappa = \tfrac{1}{2}c^2, \tag{7.8}$$

where

$$\kappa = -\frac{\mathrm{d}\alpha}{\mathrm{d}s} = -\cos\alpha\frac{\mathrm{d}\alpha}{\mathrm{d}x}. \tag{7.9}$$

So on integration with respect to x we have

$$\int_{-X}^{X} \{[\tfrac{1}{2}(u^2+v^2)_{y=\eta}-c^2]+\eta\}\,\mathrm{d}x = [\sin\alpha]_{-X}^{X}. \tag{7.10}$$

As $X\to\infty$, so $\alpha\to 0$ by hypothesis, and the right-hand side vanishes. In other words the surface tension contributes on the whole zero to the vertical momentum of the fluid. So from (7.9) and (7.12) we have

$$\int_{-X}^{X} \eta\,\mathrm{d}y \to \int_{-X}^{X} \tfrac{1}{2}[c^2-(u^2+v^2)]_{y=-Y}\,\mathrm{d}x - \int_{-Y}^{\eta} uv\,\mathrm{d}y\Big|_{-X}^{X}. \tag{7.11}$$

If we write $u' = u-c$ for the horizontal velocity in the stationary frame of reference, then when X and Y are both large we find to lowest order

$$\int_{-X}^{X} \eta\,\mathrm{d}y \to \int_{-X}^{X} (-cu')_{y=-Y}\,\mathrm{d}x - \int_{-Y}^{\eta} cv\,\mathrm{d}y\Big|_{-X}^{X}. \tag{7.12}$$

On replacing u', v by ϕ'_x and ϕ'_y respectively the right-hand side becomes

$$c[\phi'(-X,\eta)-\phi'(X,\eta)]. \tag{7.13}$$

By (7.2), this tends to the value $-cC$. So (7.3) is proved.

Physically this means that in order to maintain a non-zero net displacement at the free surface, some circulation at great depths is required. The situation is analogous to the lift on an aerofoil produced by a net circulation around the wing, except that in the present case the flow in only about half the space is involved.

However, we have seen that the vortex component of the flow vanishes, so no circulation C can be provided. Thus in (7.3) C is 0 and we have

$$M = 0. \tag{7.14}$$

In other words the net displacement of fluid at the surface is zero.

8. The energy and momentum

We may define the gravitational potential energy and the surface tension energy by

$$V_{\mathrm{G}} = \int_{-\infty}^{\infty} \tfrac{1}{2}\eta^2\,\mathrm{d}x \tag{8.1}$$

and $$V_{\mathrm{T}} = \int_{-\infty}^{\infty} (\mathrm{d}s - \mathrm{d}x) = [s-x]_{-\infty}^{\infty} \tag{8.2}$$

respectively. Also the kinetic energy in the relative motion is given by

$$T = \int_{-\infty}^{\infty} \int_{-\infty}^{\eta} \tfrac{1}{2}(\phi_x'^2 + \phi_y'^2)\,\mathrm{d}x\,\mathrm{d}y. \tag{8.3}$$

Using Green's theorem we may transform (8.3) into the form

$$T = \tfrac{1}{2}c \int \phi'\,\mathrm{d}\eta, \tag{8.4}$$

the integral being taken along the surface from $x = -\infty$ to ∞. Since ϕ' and η both vanish at $x = \pm\infty$ we may integrate by parts to obtain

$$T = -\tfrac{1}{2}c \int \eta\,\mathrm{d}\phi' = -\tfrac{1}{2}c \int \eta\,\mathrm{d}\phi \tag{8.5}$$

since $\mathrm{d}\phi' = \mathrm{d}\phi - c\,\mathrm{d}x$, and $\int \eta\,\mathrm{d}x$ vanishes by (7.14).

It can also be shown (see the Appendix) that

$$2T = \pi c D + c^2 M, \tag{8.6}$$

where D is the strength of the dipole, defined in (6.7). Since $M = 0$ this becomes simply

$$2T = \pi c D. \tag{8.7}$$

We note that as $w \to \infty$ so $\zeta \to -1$, by (3.3). So from (3.4), (6.4) and (6.5) we have

$$D = A_1 - A_2 + A_3 - \dots. \tag{8.8}$$

Thus (8.4), (8.5) and (8.7) represent three different ways of evaluating T.

We may also define the total horizontal momentum or impulse

$$I = \lim_{X,\,Y\to\infty} \int_{-X}^{X} \int_{-Y}^{\eta} (-u')\,\mathrm{d}x\,\mathrm{d}y. \tag{8.9}$$

However, for large values of X and Y we find

$$I = 2D \tan^{-1}(X/Y) + cM \tag{8.10}$$

(see the Appendix). Since $M = 0$ equation (8.10) becomes simply

$$I = 2D \tan^{-1}(X/Y). \tag{8.11}$$

For example, if we make X and Y tend to infinity in such a way that $X/Y \to \infty$ then

$$I = \pi D. \tag{8.12}$$

From (8.7) we have then $$2T = cI, \tag{8.13}$$

as for periodic waves (c.f. Hogan 1979). If on the other hand $X/Y \to 0$, i.e. we consider a large section of fluid that is deep compared to its width, then we find

$$I = 0. \tag{8.14}$$

This means that the positive momentum at higher levels is exactly compensated by the negative momentum at deeper levels. This indeterminacy in the momentum, or added mass, is typical of dipole flows.

c	V_T	V_G	V	T	E
0.9276	1.806	0.260	3.066	1.294	4.360
0.95	1.713	.291	2.974	1.294	4.268
1.00	1.505	.348	2.765	1.275	4.040
1.05	1.297	.384	1.681	1.225	2.906
1.10	1.091	.398	1.489	1.143	2.632
1.15	0.889	.391	1.280	1.031	2.311
1.20	0.695	.361	1.056	0.890	1.946
1.25	0.512	.309	0.821	0.727	1.548
1.30	0.34	.24	0.58	0.53	1.11

TABLE 4. Integral properties of solitary waves

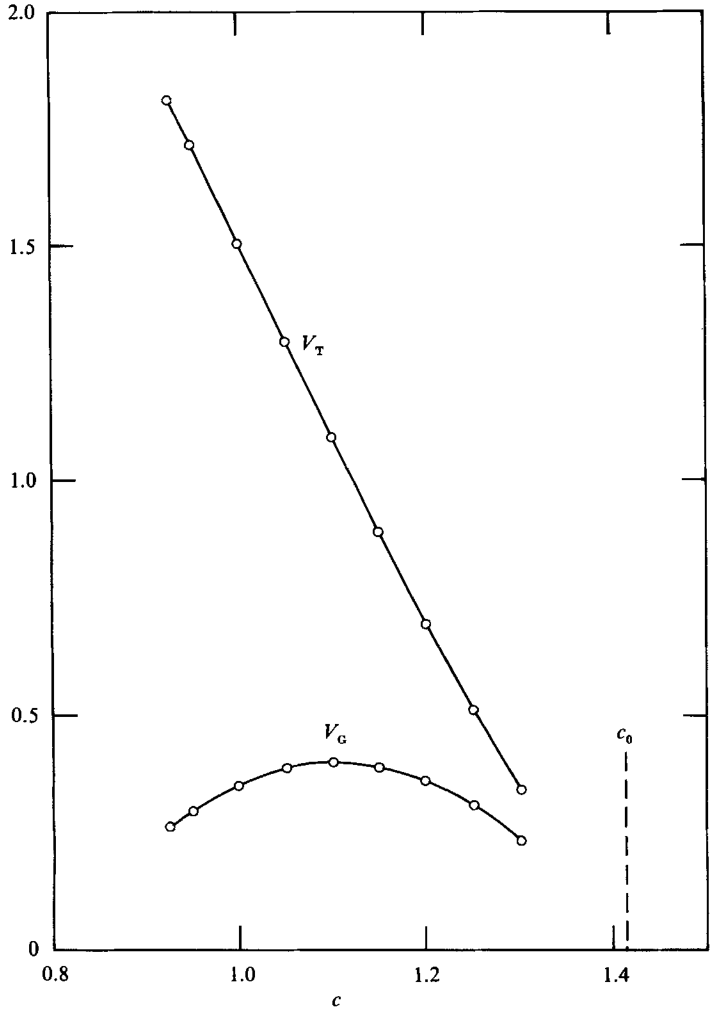

FIGURE 8. The gravitational potential energy V_G and the surface tension energy V_T as functions of the dimensionless phase speed c.

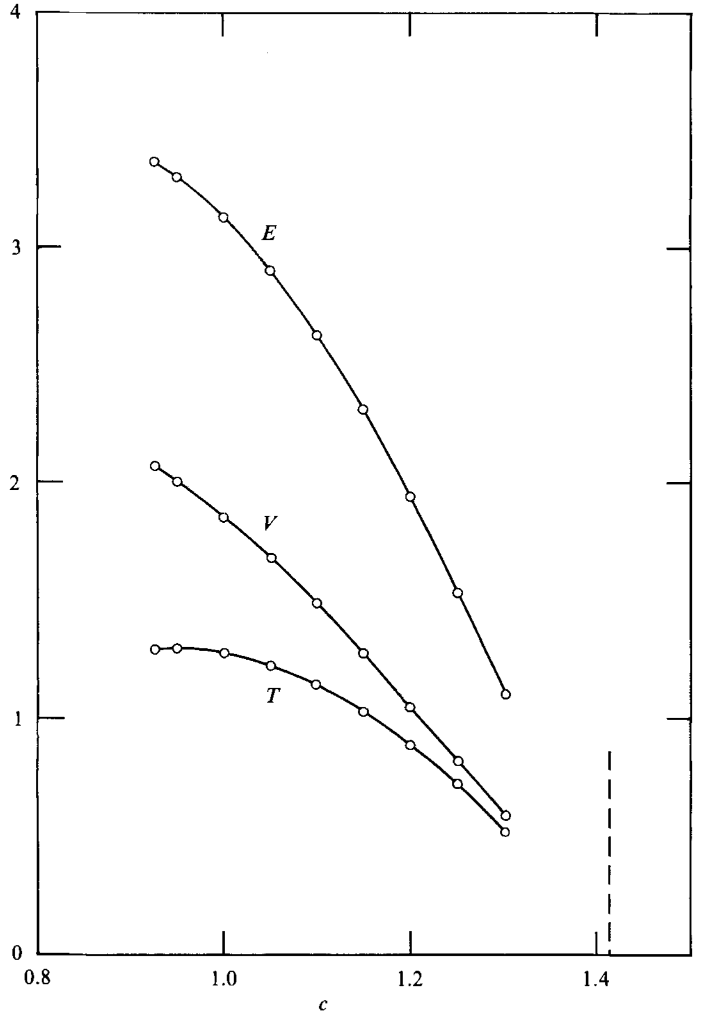

FIGURE 9. The combined potential energy V, the kinetic energy T and the total energy E as functions of the phase speed c.

The computed values of V_T, V_G and T are shown in table 4; also the combined potential energy $V = V_T + V_G$ and the total energy $E = T + V$. The two potential energies are compared in figure 8. It will be seen that V_T always exceeds V_G, so that the waves are essentially surface-tension waves, or ripples. Also V_G has a maximum in the range of c, as was found by Hogan (1980) for periodic C–G waves in deep water.

The combined potential energy V and the kinetic energy T are compared in figure 9. Both vary monotonically with c, but the potential energy always exceeds the kinetic. At the highest values of c, corresponding to the lowest values of the surface slope α_{max}, they become nearly equal.

9. Conclusions

We have found strong numerical evidence for the existence of solitary C–G waves on deep water, with phase-speeds c in the range $0.9267\,(gT)^{\frac{1}{4}} \leqslant c \leqslant 1.30\,(gT)^{\frac{1}{4}}$. This compares with the minimum speed c_0 of small-amplitude C–G waves given by $c_0 = 1.414\,(gT)^{\frac{1}{4}}$.

The solitary waves are not contiguous to the small-amplitude waves; a finite amplitude is necessary for their existence. However periodic C–G waves, with finite length, have the property that their phase speed *de*creases, in general, as their steepness is increased. Hence the solitary waves that we have found are very probably contiguous to a class of periodic C–G waves of *finite* steepness, when the wavenumber tends to 0. This might already be expected or inferred from previously published numerical calculations of C–G waves.

It may be noted that the solitary wave profiles that we have found are qualitatively similar to the *depression* solitary C–G waves in shallow water calculated by Hunter & Vanden-Broeck (1983). It is possible that our solutions are analytically contiguous to these, as the depth becomes infinite.

We shall leave the complete mapping of C–G waves, as a function of both wavenumber and maximum surface slope, to another paper. Nevertheless we can here anticipate that the solitary C–G waves that we have found will be a useful landmark in any such survey.

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Appendix. Proof of equations (8.6) and (8.10)

Let u and v denote the horizontal and vertical components of velocity in the moving frame of reference, in which the flow appears steady, and let $u' = u - c$ be the horizontal velocity in the stationary reference frame. Then we consider the kinetic energy contained in a large area bounded by the two vertical planes $x = \pm X$, the horizontal plane $y = -Y$, and the free surface $y = \eta$. By definition

$$\begin{aligned} 2T &= \iint (u'^2 + v^2)\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint [(u-c)^2 + v^2]\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint (u^2+v^2)\,\mathrm{d}x\,\mathrm{d}y - 2c\iint u\,\mathrm{d}x\,\mathrm{d}y + c^2\iint \mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{A 1}$$

In the first integral in (A 1) we have

$$\iint (u^2+v^2)\,\mathrm{d}x\,\mathrm{d}y = \left|\frac{\partial(\phi,\psi)}{\partial(x,y)}\right| \mathrm{d}x\,\mathrm{d}y = \mathrm{d}\phi\,\mathrm{d}\psi, \tag{A 2}$$

where ϕ and ψ are the velocity potential and stream function in the steady motion. Now at large distances we have

$$\phi + \mathrm{i}\psi = cz + \frac{D}{z} \tag{A 3}$$

so

$$\left.\begin{aligned} \phi &= cx + \frac{Dx}{x^2+y^2}, \\ \psi &= cy - \frac{Dy}{x^2+y^2}. \end{aligned}\right\} \tag{A 4}$$

Hence

$$\iint \mathrm{d}\phi\,\mathrm{d}\psi = 2cXcY + 2\int_{-cY}^{0} \frac{DX}{X^2+y^2}\,\mathrm{d}(cy) - \int_{-cX}^{cX} \frac{DY}{x^2+Y^2}\,\mathrm{d}(cx). \tag{A 5}$$

In the first integral we may make the substitution $y = X\tan\theta$, and in the second $x = Y\tan\theta$. Hence we find

$$\iint \mathrm{d}\phi\,\mathrm{d}\psi = 2c^2XY + cD[\pi - 4\tan^{-1}(X/Y)]. \tag{A 6}$$

Now in the second integral in (A 1) we have

$$\int_{-Y}^{\eta} u\,\mathrm{d}y = \psi_{y=\eta} - \psi_{y=-Y} = cY - \frac{DY}{x^2+Y^2}. \tag{A 7}$$

Hence

$$\int_{-X}^{X}\int_{-Y}^{\eta} u\,\mathrm{d}y\,\mathrm{d}x = 2cXY - 2D\tan^{-1}(X/Y). \tag{A 8}$$

Lastly, for the third integral in (A 1) we have

$$\iint \mathrm{d}x\,\mathrm{d}y = 2XY + M. \tag{A 9}$$

Therefore altogether (A 1) gives

$$2T = \pi cD + c^2 M \tag{A 10}$$

as stated.

We note that the total momentum $-I$ of the relative motion contained within the specified area is given by

$$-I = \int_{-X}^{X}\int_{-Y}^{\eta} u'\,\mathrm{d}x\,\mathrm{d}y = \int_{-X}^{X}\int_{-Y}^{\eta} (u-c)\,\mathrm{d}x\,\mathrm{d}y \tag{A 11}$$

so that from (A 8)

$$I = 2D\tan^{-1}(X/Y) + cM \tag{A 12}$$

as was to be proved.

REFERENCES

BENJAMIN, T. B. 1967 Internal waves of permanent form in fluids of great depth. *J. Fluid Mech.* **29**, 559–592.

BLOOR, M. I. G. 1978 Large amplitude surface waves. *J. Fluid Mech.* **84**, 167–179.

BOUSSINESQ, J. 1872 Théorie des ondes et des ramous qui se propagent le long d'un canal rectangulaire horizontal. *J. Maths Pure Appl.* **17**, 55–108.

CHEN, B. & SAFFMAN, P. G. 1979 Steady gravity–capillary waves on deep water II. Numerical results for finite amplitude. *Stud. Appl. Maths* **62**, 95–111.

CHOU, V. H. & MEI, C. C. 1970 On slowly varying Stokes waves. *J. Fluid Mech.* **41**, 873–887.

CHOU, V. H. & MEI, C. C. 1971 The non-linear evolution of Stokes waves in deep water. *J. Fluid Mech.* **47**, 337–351.

CRAPPER, G. D. 1957 An exact solution for progressive capillary waves of arbitrary amplitude. *J. Fluid Mech.* **96**, 417–445.

DAVIS, R. E. & ACRIVOS, A. 1967 Solitary internal waves in deep water. *J. Fluid Mech.* **29**, 593–607.

HOGAN, S. J. 1979 Some effects of surface tension on steep water waves. *J. Fluid Mech.* **91**, 167–180.

HOGAN, S. J. 1980 Some effects of surface tension on steep water waves. Part 2. *J. Fluid Mech.* **96**, 417–445.

HOGAN, S. J. 1981 Some effects of surface tension on steep water waves. Part 3. *J. Fluid Mech.* **110**, 381–410.

HUNTER, J. K. & VANDEN-BROECK, J.-M. 1983 Solitary and periodic gravity–capillary waves of finite amplitude. *J. Fluid Mech.* **134**, 205–219.

KORTEWEG, D. J. & DE VRIES, G. 1895 On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *Phil. Mag.* **39**, 422–433.

LAMB, H. 1932 Hydrodynamics, 6th edn. Cambridge University Press.

LONGUET-HIGGINS, M. S. 1974 On the mass, momentum, energy and circulation of a solitary wave. *Proc. R. Soc. Lond.* A **337**, 1–13.

LONGUET-HIGGINS, M. S. 1988 Limiting forms for capillary–gravity waves. *J. Fluid Mech.* **194**, 351–375.

MILES, J. W. 1980 Solitary waves. *Ann. Rev. Fluid Mech.* **12**, 11–43.

RAYLEIGH, LORD 1976 On waves. *Phil. Mag.* **1**, 257–279.

RUSSELL, J. S. 1838 Report of the Committee on Waves. *Rep. Meet. Brit. Assoc. Adv. Sci., Liverpool 1837*, pp. 417–496. London: John Murray.

SCHOOLEY, A. H. 1958 Profiles of wind-created water waves in the capillary–gravity transition region. *J. Mar. Res.* **16**, 100–108.

SCHWARTZ, L. W. & FENTON, J. D. 1982 Strong nonlinear waves. *Ann. Rev. Fluid Mech.* **14**, 39–60.

SCHWARTZ, L. W. & VANDEN-BROECK, J.-M. 1979 Numerical solution of the exact equations for capillary–gravity waves. *J. Fluid Mech.* **95**, 119–139.

WILTON, J. R. 1915 On ripples. *Phil. Mag.* **29** (6), 688–700.

YUEN, H. C. & LAKE, B. M. 1975 Nonlinear deep water waves: theory and experiment. *Phys. Fluids* **18**, 956–960.